Communication and Concurrency
Lecture 8

Colin Stirling (cps)

School of Informatics

17th October 2011

Bisimulation equivalence

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We deal first with conditions $1-4$

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  Proof: Since $E \sim F$, $(E, F) \in B_1$ for some bisimulation B_1. Since $F \sim G$, $(F, G) \in B_2$ for some bisimulation B_2. So $(E, G) \in B_1 \circ B_2$. We show that $B_1 \circ B_2$ is a bisimulation.

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4. $E[f] \sim F[f]$
5. $E \backslash K \sim F \backslash K$

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- $G \xrightarrow{a} G'$ and $E' = E$. So $F \mid G \xrightarrow{a} F \mid G'$, and by definition $((E \mid G'), (F \mid G')) \in B$.

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- $G \xrightarrow{a} G'$ and $E' = E$. So $F \mid G \xrightarrow{a} F \mid G'$, and by definition $((E \mid G'), (F \mid G')) \in B$.
- $a = \tau$ and $E \xrightarrow{b} E'$ and $G \xrightarrow{\overline{b}} G'$. $F \xrightarrow{b} F'$ for some F' such that $E' \sim F'$, so $F \mid G \xrightarrow{\tau} F' \mid G'$, and therefore $((E' \mid G'), (F' \mid G')) \in B$.

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Symmetrically for a transition $F \mid G \xrightarrow{a} F' \mid G'$.

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  For any G and H, if $G \sim H$, then $G \models \Phi$ iff $H \models \Phi$.

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- Basis: $\Phi = \texttt{tt}$ or $\Phi = \texttt{ff}$. Clear.
- Step: We consider only the case $\Phi = [K]\Psi$. By symmetry, it suffices to show that $G \models [K]\Psi$ implies $H \models [K]\Psi$.
  Assume $G \models [K]\Psi$. For any G' such that $G \stackrel{a}{\longrightarrow} G'$ and $a \in K$, it follows that $G' \models \Psi$.
  Let $H \stackrel{a}{\longrightarrow} H'$ (with $a \in K$). Since $G \sim H$, there is a G' such that $G \stackrel{a}{\longrightarrow} G'$ and $G' \sim H'$. By the induction hypothesis $H' \models \Psi$, and therefore $H \models \Phi$.

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- E is image-finite if all processes reachable from it are immediately image-finite.

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  Need to show $H \xrightarrow{a} H_i$ and $G' \equiv_{\mathrm{HM}} H_i$
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  (Here we use the fact that M is closed under complement.)
- Let $\Psi = \Phi_1 \wedge \ldots \wedge \Phi_n$.
  $G \models \langle a \rangle \Psi$ but $H \not\models \langle a \rangle \Psi$ because each H_i fails to have property Ψ. Contradicts $G \equiv_{\mathrm{HM}} H$.

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- Case $H \xrightarrow{a} H'$ is symmetric.

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- if $F \stackrel{a}{\Longrightarrow} F'$ then $E \stackrel{a}{\Longrightarrow} E'$ for some E' such that $(E', F') \in B$
- Two processes E and F are weak bisimulation equivalent (or weakly bisimilar) if there is a weak bisimulation relation B such that $(E, F) \in B$. We write $E \approx F$ if E and F are weakly bisimilar

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- Two observationally image finite processes are weakly bisimilar iff they satisfy the same properties of observational Hennessy-Milner logic.

Exercise

Which of the following are weakly bisimilar?

		Y/N
$a.\tau.b.0$	$a.b.0$	
$a.(b.0 + \tau.c.0)$	$a.(b.0 + c.0)$	
$a.(b.0 + \tau.c.0)$	$a.(b.0 + \tau.c.0) + a.c.0$	
$a.0 + b.0 + \tau.b.0$	$a.0 + \tau.b.0$	
$a.0 + b.0 + \tau.b.0$	$a.0 + b.0$	
$a.(b.0 + \tau.b.0)$	$a.b.0$	